

Reg. No. :

Name :

III Semester M.Sc. Degree (CBSS – Reg./Suppl./Imp.)

Examination, October 2021

(2018 Admission Onwards)

MATHEMATICS

MAT3C14 : Advanced Real Analysis

Max. Marks : 80

Time : 3 Hours

PART – A

Answer four questions from this Part. Each question carries 4 marks.

1. Define uniform bounded functions and give an example.
2. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

3. Consider $f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2x}$. For what values of x does the series converge absolutely?

4. Show that $\lim_{x \rightarrow +\infty} x^n e^{-x} = 0$ for every n .

5. Show that $\log \Gamma$ is convex on $(0, \infty)$.

6. Find $\lim_{n \rightarrow \infty} \frac{n}{\log n} \left[n^{\frac{1}{n}} - 1 \right]$. (4×4=16)

PART – B

Answer 4 questions from this Part without omitting any Unit. Each question carries 16 marks.

Unit – I

7. a) Compare pointwise convergence and uniform convergence.
b) If $\{f_n\}$ is a sequence of continuous function on E and if $f_n \rightarrow f$ uniformly on E , then show that f is continuous on E .

8. a) Suppose K is compact, and

- i) $\{f_n\}$ is a sequence of continuous functions on K .
- ii) $\{f_n\}$ converges pointwise to a continuous function f on K .
- iii) $f_n(x) \geq f_{n+1}(x)$ for all $x \in K$, $n = 1, 2, \dots$

Then show that $f_n \rightarrow f$ uniformly on K .

b) Let α be monotonically increasing on $[a, b]$. Suppose $f_n \in \mathcal{R}(\alpha)$ on $[a, b]$, $n = 1, 2, \dots$ and suppose $f_n \rightarrow f$ uniformly on $[a, b]$ then prove that $f \in \mathcal{R}(\alpha)$ on $[a, b]$ and $\int_a^b f d\alpha = \lim_{n \rightarrow \infty} \int_a^b f_n d\alpha$.

9. a) Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a, b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a, b]$. If $\{f'_n\}$ converges uniformly on $[a, b]$ then show that $\{f_n\}$ converges uniformly on $[a, b]$, to a function and $f'(x) = \lim_{n \rightarrow \infty} f'_n(x)$ ($a \leq x \leq b$).

b) Define algebra and give an example.

Unit – II

10. a) State and prove Taylor's theorem.

b) Suppose a_0, \dots, a_n are complex numbers $n \geq 1$, $a_n \neq 0$, $P(z) = \sum_{k=0}^{\infty} a_k z^k$. Then prove that $P(z) = 0$ for some complex number z .

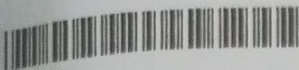
11. a) Define orthogonal system of functions.

b) If $\{\phi_n\}$ be orthonormal on $[a, b]$ and if $f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x)$, then prove that $\sum_{n=1}^{\infty} |c_n|^2 \leq \int_a^b |f(x)|^2 dx$.

c) If $f(x) = 0$ for all x in some segment J , then prove that $\lim S_N(f; x) = 0$ for every $x \in J$.

12. a) Define beta function.

b) State and prove Stirling's formula.



Unit – III

13. a) Show that a linear operator A on a finite-dimensional vector space X is one-to-one if and only if the range of A is all of X .
- b) If $A, B \in L(\mathbb{R}^n, \mathbb{R}^m)$ and c is a scalar, then prove that $\|A + B\| \leq \|A\| + \|B\|$, $\|cA\| = |c|\|A\|$. With the distance between A and B is defined as $\|A - B\|$, prove that $L(\mathbb{R}^n, \mathbb{R}^m)$ is a metric space.
- c) Let Ω be the set of all invertible linear operators on \mathbb{R}^n . Then prove that Ω is an open subset of $L(\mathbb{R}^n)$ and the mapping $A \rightarrow A^{-1}$ is continuous on Ω .
14. a) Suppose f maps an open set $E \subset \mathbb{R}^n$ into \mathbb{R}^m and f is differentiable at a point $x \in E$. Then prove that the partial derivatives $(D_j f_i)(x)$ exist and $f'(x)e_j = \sum_{i=1}^m (D_j f_i)(x)u_i$ ($1 \leq j \leq n$), where $\{e_1, \dots, e_n\}$ and $\{u_1, \dots, u_m\}$ are the standard bases of \mathbb{R}^n and \mathbb{R}^m .
- b) Suppose f maps a convex open set $E \subset \mathbb{R}^n$ into \mathbb{R}^m , f is differentiable in E and there is a real number M such that $\|f'(x)\| \leq M$ for every $x \in E$. Then prove that $|f(b) - f(a)| \leq M |b - a|$ for all $a \in E, b \in E$.
15. State and prove inverse function theorem. (4×16=64)